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A handy analytical approximate solution for the magnetohydrodynamic flow of blood in a porous channel

Una solución aproximada y analítica del flujo magnetohidrodinámico de la sangre en un canal poroso

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Abstract

This work presents a new version of the Picard method, known as the boundary values problems Picard method (BVPP), to obtain an analytical approximate solution for a highly complex nonlinear differential equation that models the magnetohydrodynamic flow of blood through a porous channel. The proposed method is versatile and can produce compact and easily evaluated analytical expressions that accurately capture the scientific phenomena being studied, making it ideal for practical applications. BVPP transforms a differential equation into an integral equation and utilizes an iterative algorithm like that of the basic Picard method. However, unlike the basic method, BVPP allows for the selection of an appropriate initial function and involves several adjustable parameters that can be optimized to obtain a precise analytical approximate solution with minimal effort. Overall, BVPP represents a significant advancement in the analysis of complex nonlinear differential equations, particularly in the field of biomedical engineering.

Keywords: Approximate solution; nonlinear ordinary differential equation; Picard method; flow of blood through a porous channel.

Resumen

Este trabajo presenta una versión nueva del método de Picard, conocido como método de Picard para problemas de valores en la frontera (BVPP, por sus siglas en inglés), para obtener una solución analítica aproximada para la ecuación diferencial no lineal difícil de resolver que modela el flujo magnetohidrodinámico de la sangre a través de un canal poroso. El método propuesto es versátil y puede proporcionar expresiones analíticas compactas, fáciles de evaluar, que describen con precisión los fenómenos científicos estudiados, haciendo a BVPP un método ideal para usarse en aplicaciones prácticas. BVPP transforma una ecuación diferencial en una ecuación integral y utiliza un algoritmo iterativo, tal como en el método de Picard básico; sin embargo, a diferencia del método básico, BVPP permite la elección de una función inicial apropiada provista de varios parámetros de ajuste que se optimizan para obtener una solución analítica aproximada y precisa con un esfuerzo mínimo. En términos generales, BVPP representa un avance significativo en el análisis de ecuaciones diferenciales difíciles de resolver, particularmente en el campo de la ingeniería biomédica.

Palabras clave: Solución aproximada; ecuación diferencial no lineal; método de Picard; flujo de sangre a través de un canal poroso.

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Introduction

Modelling nature processes in the mathematical realm is a difficult task because most of these processes are nonlinear, as a result, we need to use complex mathematical models to find their approximate solutions; however, these models may not always yield the desired outcomes. On the other hand, the use of differential equations has proven to be a valuable tool when seeking to obtain meaningful results. The dynamics of natural phenomena, ranging from physical processes to biological interactions, necessitate a deep understanding to be captured and analysed mathematically. In response to these challenges, researchers continually propose innovative methods to get exact and approximate solutions to the differential equations that govern these problems; unfortunately, the search for such solutions is not always an easy task, hence, the need for this research.

As it is well known, finding the solutions for linear differential equations is a relevant subject. In particular, the theory of these equations and their methods of solution can be found in many texts of differential equations (Boyce & DiPrima, 2012; Simmons, 2016; Zill, 2012). Unlike linear ordinary differential equations (ODES), the case of nonlinear ordinary differential equations with exact solutions is less frequent (Boyce & DiPrima, 2012; Simmons, 2016; Zill, 2012). The present work will show the potential of a proposed method to find handy approximate solutions for a highly nonlinear case study with relative ease. In fact, the diversity of nonlinear problems has led to the proposal of several alternative methods aiming to solve various types of nonlinear differential equations.

According to the literature, some of the most employed methods are: variational approaches (Assas, 2007), tanh method (Evans & Raslan, 2005), exp-function (Mahmoudi *et al*., 2008; Xu, 2007), Adomian's decomposition method (Adomian, 1988; Babolian & Biazar, 2002), parameter expansion (Mahmoudi *et al*., 2008), homotopy perturbation method (HPM) (Adamu & Ogenyi, 2017; Beléndez *et al*., 2008; El-Dib, 2017; El-Dib & Moatimid, 2018; Filobello-Nino *et al*., 2016b; He, 2000, 2006, 2008; Tripathi & Mishra, 2016; Vazquez-Leal *et al*., 2012), perturbation method (Filobello-Nino *et al*., 2013; Holmes, 2013), modified Taylor series method (Vazquez-Leal *et al*., 2015), Picard method (Filobello-Nino *et al*., 2016a), PSEM method (Filobello-Nino *et al*., 2020), homotopy analysis method (Patel *et al*., 2012; Sami *et al*., 2008; Shijun, 1998), variational iteration method (He & Wu, 2007), homotopy asymptotic method (Marinca & Herisanu, 2011), among others.

The aim of this work is to propose a novel modified version of the Picard method, the boundary values problems Picard method (BVPP) (Filobello-Nino *et al*., 2016a), aiming to provide an analytical approximate solution for the relevant highly nonlinear differential equation that describes the magnetohydrodynamic flow of blood through a porous channel (Misra *et al*., 2011).

The utilization of magnetohydrodynamics extends across various applications, including its application in studying the flow of arterial blood under the influence of an applied magnetic field. In this context, we will see that the BVPP method is able to provide analytical approximate solutions even for the case of highly nonlinear differential equations defined in closed intervals. This is particularly significant as many investigations in this field primarily rely on numerical approaches, emphasizing the practicality and versatility of the introduced method.

In brief, BVPP is expressed as follows. Given an ordinary differential equation, the method expresses it as an integral equation; then, an iterative process is introduced as it occurs with the basic Picard method. Nevertheless, unlike Picard method, BVPP will employ its freedom to propose a trial function which contains one or more parameters to be determined for the same method. Although there are several options to optimally determine the values of the above-mentioned unknown parameters, this work will use the numerical solution in order to obtain a handy analytical approximate solution for the relevant proposed problem mentioned above. The precision of the obtained results will show the potential of BVPP for future applications.

Materials and methods

Basic idea for the problem of a magnetohydrodynamic flow of blood in a porous channel

Magntohydrodynamic (MHD) is the part of mechanics concerned with the study of moving electrically conductive fluids in the presence of magnetic and electric fields. The importance of the MHD lies on providing several possibilities of application such as in metallurgy, electricity generation, and electromagnetic boosters. In the same way, MHD is used to study the flow of arterial blood under the action of an applied magnetic field (Misra *et al*., 2011). In the medical field, these investigations are of great value in the treatment of cardiovascular disorders, considering the case of accelerated circulation. Given that blood erythrocytes have small negative charges associated to them, it is expected that the application of a magnetic field can influence the movement of the erythrocytes and, as a consequence, that the flow of blood can possibly be affected too.

In effect, the behaviour of the blood flow subjected to different situations has been reported by several authors. For example, Misra & Shit (2007) proposed a model in order to study the blood flow through a stenosed arterial segment considering the slip velocity at the wall of the artery. Fung & Sobin (1969) reported their investigation about the flow of blood between two endothelial layers, and the mass transference problem in an alveolar sheet was studied by Fung & Tang (1975). In the same way, Misra *et al*. (2008) and Misra & Shit (2009) mathematically modelled the blood flow in a channel with stretching walls, considering the blood as a non-Newtonian fluid. On the other hand, Fukada & Kaibara (1980) noted that under some conditions blood could exhibit viscoelastic behaviour due to the viscoelastic properties of the erythrocytes.

This paper is dedicated to get an analytical approximate solution for the MHD boundary layer flow of blood in the aortic arch (Misra *et al*., 2011). The flow is assumed to obey Walter's liquid-B model, and this supposition results in a highly nonlinear partial differential equation (NPDE). Nevertheless, after introducing some non-dimensional quantities, it is possible to express the above mentioned NPDE in terms of a complicated and long nonlinear ordinary differential equation (NODE). Despite the difficulty of the nonlinear ODE to be solved, this work will provide a handy analytical approximate solution that is ideal for practical applications.

Figure 1 shows the physical model of a steady blood flow in a channel provided with porous boundaries bounded by two thin parallel plates which is under the effect of a transverse magnetic field. The erythrocytes (red blood cells) are the particles influenced by the magnetic field because they have a small negative charge. The x-axis is chosen along the centre line of the channel, and the y-axis is in the transverse direction. In the same way, the flow is assumed to be symmetric about x-axis, and the porous walls are represented by the values $y = R/2$ and $y = -R/2$, where R is the channel width. The fluid injection (extraction) is given through the porous with velocity $V/2$ ($V > 0$ suction and $V < O$ injection).

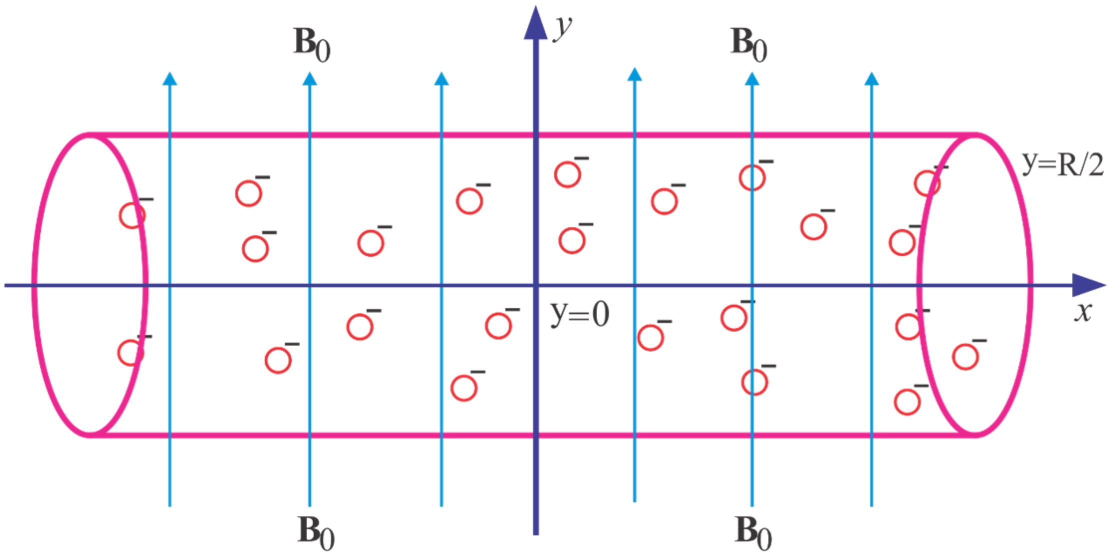

Figure 1. Figure shows the physical model of a porous channel through which blood flows under the action of a magnetic transverse field.
Source: Author's own elaboration.

Let u, v be the x component and y component of the velocity; Bo is the strength of the applied magnetic field; and ρ, μ, σ, Ko are respectively the density, kinematic viscosity, electrical conductivity, and coefficient of viscoelasticity of the fluid. Therefore, the MHD boundary layer flows for an incompressible fluid (Walter's Liquid-B fluid) in terms of the non-dimensional variables:

$$\xi = \frac{x}{R}, \quad \eta = \frac{y}{R}, \quad u = V\xi f'(\eta), \quad v = -Vf(\eta) \qquad (1)$$

is given by the nonlinear equation:

$$f''' - Mf' + Re(ff'' - f'^2) = K_1[2f'f''' - ff^{iv} - f''^2] \qquad (2)$$

where $Re = \frac{VR}{v}$ is the Reynolds number, $M = Ra^2 = \frac{\sigma B_0^2 R^2}{v}$ denotes the square of the Hartmann number, and the viscoelastic parameter is given by $K_1 = \frac{VK_0}{Rv}$.

In terms of the non-dimensional distance η (from equation 1), the boundary conditions for equation 2 are expressed as:

$$f(\eta = 0) = 0, f''(\eta = 0) = 0, f'(\eta = 1/2) = 0, f(\eta = 1/2) = -1/2 \qquad (3)$$

Therefore, the problem consists in approximating equation 2 subjected to boundary conditions (equation 3).

Basic idea of the novel boundary value problems Picard method (BVPP)

The basic idea of Picard iteration method (PIM) consists in expressing the problem:

$$y'(t) = f(t, y(t)); \qquad y(t_0) = A \qquad (4)$$

where, in accordance with the local existence and uniqueness Picard's theorem, $y(t)$ is a function whose graph passes through (t_0, A) and satisfies $y' = f(t, y)$ in some neighbourhood of t_0, in terms of the following integral equation derived from equation 4 (Elsgolts, 1977; Filobello-Nino et al., 2016a; Simmons, 2016; Zill, 2012):

$$y(t) = A + \int_{t_0}^{t} f(t', y(t'))dt' \qquad (5)$$

As a matter of fact, PIM expresses the solution of equation 5 as the limit of a sequence of functions $y_n(t)$ $(n \rightarrow \infty)$ defined through the recurrence formula:

$$y_n(t) = A + \int_{t_0}^{t} f(t', y_{n-1}(t'))dt', \qquad n = 1,2 \qquad (6)$$

A relevant point about equation 6 occurs when $f(t, y)$ is continuous in all its arguments and satisfies a Lipschitz condition in the y variable on a band defined by $a \leq x \leq b$ and $-\infty < y < \infty$. If the above conditions are satisfied, then let (x_0, y_0) be any point of the above mentioned band, then the initial value problem (equation 4) has one and only one solution in $a \leq x \leq b$. In a sequence, it is known that independently of the selected initial function $y_0(t)$, the sequence $y_n(t)$ caused by the iterative procedure (equation 6) converges to the solution of equation 4 in the above mentioned interval. This convergence highlights the effectiveness of the Picard method for solving differential equations, with a more in-depth discussion available in works like those of Filobello-Nino *et al.* (2016a) and Simmons (2016), the latter being an essential resource for a comprehensive understanding of the method.

Next, we focus on the case of boundary value problems (BVP) with Dirichlet boundary conditions, assuming that the values of the sought solution are given at two points t_0 and t_1. Therefore, let us consider the following problem:

$$y''(t) = f(t, y(t), y'(t)); \qquad y(t_0) = A, \qquad y(t_1) = C \qquad (7)$$

To begin, let us approach the problem, assuming that we know the value of $y'(t_0)$ (although it is unknown), and that the right hand side of equation 7 is a continuous function. On the other hand, the freedom of BVPP method is employed with the purpose of choosing an adequate trial function $y_0(t)$, aiming to include the two boundary values and to accelerate the convergence of the procedure (Filobello-Nino *et al.*, 2016a). BVPP method proposes as trial function a polynomial function $P(t)$, which contains one or more adjusting parameters $D, E, F, \dots$ to be determined, that is:

$$y_0(t) = P(t, D, E, F, \dots) \qquad (8)$$

In accordance to equation 7, BVPP employs the following integral equation:

$$y(t) = A + \beta t + \int_{t_0}^{t}\int_{t_0}^{t} f[t', y(t'), y'(t')]dt'dt \qquad (9)$$

where, as it was already mentioned, the value of $y'(t_0) = \beta$ is unknown for the time being, and it is determined by the BVPP alghorithm.

The solution for equation 9 can be expressed as the limit of a sequence of functions $y_n(t)$ $(n \to \infty)$, in accordance to the following recurrence formula:

$$\begin{aligned} y_n(t,\beta,D,E,F,..) &= A+\beta t \\ &+ \int_{t_0}^{t}\int_{t_0}^{t} f[t', y_{n-1}(t',D,E,F,..), y'_{n-1}(t',D,E,F,..)]dt'dt \end{aligned} \quad (10)$$

Assuming that $f(t,y,y')$ is continous in all its arguments and also that it satisfies a Lipschitz condition in y and y' variables, in a neighbourhood of the point (t_0, y_0, y'_0), then, irrespective of equation 8, the successive approximations $y_n(t)$ which follow from equation 10 converge to a solution of the following problem, which resembles equation 7 (Filobello-Nino *et al*., 2016a; Simmons, 2016).

$$y''(t) = f(t, y(t), y'(t)); \qquad y(t_0) = A, \qquad y'(t_0) = \beta \qquad (11)$$

in a small neighbourhood of the point t_0.

Nevertheless, sometimes it occurs that $f(t,y,y')$ satisfies a Lipschitz condition in the band $t_0 \le t \le t_1$, $-\infty \le y \le \infty$, and $-\infty \le y' \le \infty$. In this case, if (t_0, y_0, y'_0) is any point into the above mentioned band, then the equation 11 has one and only one solution: $y = y(t)$ in $t_0 \le t \le t_1$ (Simmons, 2016).

Next, in order to ensure that the $n-th$ iteration of BVPP (equation 10) is also an approximate solution for equation 7, the values of $\beta, D, E, F, \ldots$ are chosen in order to guarantee that the approximate solution satisfies $y(t_1) = C$ and, for the same reason, equation 7. Although equation 7 and equation 11 are related in this way, in practice it is not necessary to explicitly consider the auxiliary equation 11. There is still the question of calculating the earlier mentioned parameters. Although Filobello-Nino *et al*. (2016a) indicate three manners to optimally calculate their values, in this work we will only employ the first of them. The basic rudiments of this procedure, denominated method 1, are as follows (Filobello-Nino *et al*., 2016a).

In practical situations, it is assumed that the nth approximation (for some n) is sufficient, then we will symbolically write equation 10 as follows:

$$y_n = H(t,\beta,D,E,F,..) \qquad (12)$$

where $H(t,\beta,D,E,F,..)$ represents the function obtained from the mentioned iterative process.

This method assumes as known, for instance, the numerical solution of ecuation 7 or somehow a known set of points of the sought solution curve (such as it will occur later in this work); hence, equation 12 is evaluated at as many points within the interval $[t_0, t_1]$ as parameters to be determined:

$$\begin{aligned} &y_n(t_0) = H(t_0,\beta,D,E,F,..), \\ &y_n(t_1) = H(t_1,\beta,D,E,F,..), \\ &y_n(t_2) = H(t_2,\beta,D,E,F,..), \\ &y_n(t_3) = H(t_3,\beta,D,E,F,..), \\ &y_n(t_4) = H(t_4,\beta,D,E,F,..), \\ &\ldots \end{aligned} \qquad (13)$$

where $t_2, t_3, t_4 \in (t_0, t_1)$ and the values of $y_n(t_0)$, $y_n(t_1)$, $y_n(t_2)$, $y_n(t_3)$, $y_n(t_4)$, ... are assumed to be known.

Equation 13 is a system of algebraic equations, whose solution allows to know the value of the parameters $\beta, D, E, F, \ldots$ It is expected that the vectorial function F of the component equations of the system (equation 13) $F: R^n \rightarrow R^n(\beta, D, E, F) \rightarrow (H(t_0, \beta, D, E, F, \ldots) - y_n(t_0), \ldots, H(t_n, \beta, D, E, F, \ldots) - y_n(t_n))$ is of class $C^1(R^n)$ and its Jacobian matrix is not zero at any point in order to guarantee the existence of a unique solution of equation 13. In addition, it is noted that equation 12 should provide a good approximation, after considering several inner points, because it is ensured that it will pass through n-points of the exact solution.

On the other hand, the above mentioned procedure for BVPP can be extended for the case of nonlinear differential equations of order greater than two. In this case, the boundary conditions of the problem will not only specify special values of the sought solution, as it occurs in equation 7, but in the same way they will contain derivative values evaluated in some points of the problem domain. In this case, we have to add other equations to the system of algebraic equations 13, obtained after differentiating equation 12, in order to guarantee that the approximate solution (equation 12) satisfies all the boundary conditions of the proposed problem.

Results

Application of boundary value problems Picard method (BVPP) to obtain an explicit analytic approximate solution for the magnetohydrodynamic flow of blood in a porous channel

In this section, we will get an explicit analytical approximate solution for the problem afore explained, emphasizing the ease with which a handy and precise analytical approximated solution is obtained for the highly nonlinear problem (equations 2 and 3), despite the difficulty of the nonlinear differential equation 2.

To start, equation 2 is rewritten in terms of the following integral equation:

$$f = A\eta + \int_0^{\eta} \int_0^{\eta_2} \int_0^{\eta_1} \left[Mf' - Re(ff'' - f'^2) + K_1\left(2f'f''' - ff^{iv} - f''^2\right)\right] d\eta_1 d\eta_2 d\eta \qquad (14)$$

where A is the unknown value of $f'(\eta = 0)$.

In accordance to the proposed method, the following iterative process is introduced from equation 14:

$$f_n = A\eta + \int_0^{\eta} \int_0^{\eta_2} \int_0^{\eta_1} [Mf'_{n-1} \quad -Re\left(f_{n-1}f''_{n-1} - {f'_{n-1}}^2\right) \\ +K_1\left(2f'_{n-1}f'''_{n-1} - f_{n-1}{f_{n-1}}^{iv} - {f''_{n-1}}^2\right)] d\eta_1 d\eta_2 d\eta, \qquad (15)$$

$n = 1,2,3,4.\ldots$

The first iteration for equation 15 corresponds to $n = 1$:

$$f_1 = A\eta + \int_0^{\eta} \int_0^{\eta_2} \int_0^{\eta_1} \left[Mf'_0 - Re\left(f_0 f''_0 - {f'_0}^2\right) + K_1\left(2f'_0 f'''_0 - f_0 {f_0}^{iv} - {f''_0}^2\right)\right] d\eta_1 d\eta_2 d\eta \qquad (16)$$

Next, taking advantage of the flexibility of the BVPP method, the following polynomial is proposed as the initial approximation:

$$f_0(\eta) = B + C\eta + D\eta^2 + E\eta^3 + F\eta^4 \qquad (17)$$

Therefore, after substituting equation 17 into equation 16 and performing elementary integrations, we get:

$$f_1(\eta) = A\eta + \frac{1}{126}\,ReF^2\eta^9 + \frac{1}{56}\,ReEF\eta^8 + \frac{1}{7}\left[-\frac{1}{30}\,Re(-2\,DF - 3\,E^2) + \frac{4}{5}\,K_1\,F^2\right]\eta^7 + \frac{1}{6}\left[\frac{1}{5}\,MF + \frac{6}{5}\,K_1\,EF - \frac{1}{20}\,Re(4\,CF - 4\,DE)\right]\eta^6 + \frac{1}{5}\left[\frac{1}{4}\,ME - \frac{1}{12}\,Re(12\,BF - 2\,D^2) + 2\,K_1\,D\,F\right]\eta^5 + \frac{1}{4}\left[\frac{1}{3}\,MD - \frac{1}{6}\,Re(6\,BE - 2\,CD) + 4\,K1\,CF\right]\eta^4 + \frac{1}{3}\left[\frac{1}{2}\,MC + \frac{1}{2}\,K_1\,(-24\,BF + 12\,CE - 4\,D^2) - \frac{1}{5}\,Re(2\,BD - C^2)\right]\eta^3 \quad (18)$$

Two solutions are provided from equation 18 for the two set of parameters M, Re, and K_1. As a matter of fact, in accordance with the proposed method, the numerical solution of equations 2 and 3 is employed in order to set up a nonlinear system to calculate the values of A, B, C, D, E and F.

The first case study assumes the following values of the parameters (Vazquez-Leal, 2020):

$$M = 2, \qquad Re = 5, \qquad K_1 = 0.005 \qquad (19)$$

From the numerical solution of this problem, for this set of parameters (equation 19) (Vazquez-Leal, 2020), we obtain the following points:

$$\begin{array}{l} (0.1, -0.120350) \\ (0.2, -0.235817) \\ (0.3, -0.341407) \\ (0.4, -0.431864) \\ (0.5, -0.50000) \end{array} \qquad (20)$$

After substituting the points from equation 20 into equation 18, and also into the necessary derivatives of equation 18, we get a system of nonlinear algebraic equations for the unknown values A, B, C, D, E and F, whose solution results in the following approximate solution.

$$y(\eta) = -\frac{4557}{3698}\,\eta + \frac{100567}{47}\,\eta^9 - \frac{198432}{59}\,\eta^8 + \frac{141967}{58}\,\eta^7 - \frac{239182}{201}\,\eta^6 + \frac{60142}{149}\,\eta^5 - \frac{16912}{209}\,\eta^4 + \frac{9683}{1224}\,\eta^3 \quad (21)$$

Table 1 compares the exact solution (Vazquez-Leal, 2020), the approximated solution proposed for the novel continuum-cancelation Leal method (CCLM) (Vazquez-Leal, 2020) for the same problem with parameters (equation 19), and the BVPP solution (equation 21). It is worth noting that equation 21 provides not only a handy solution but an accurate one (see Discussion section).

Table 1. Comparison between approximate solution (equation 21), exact solution, and another reported approximate solution.

η	Exact solution (Vazquez-Leal, 2020)	BVPP (equation 21), this work	CCLM (Vazquez-Leal, 2020) (equation 17)
0.05	-0.060479	- 0.061021	-0.060470
0.10	-0.120350	- 0.120349	-0.120333
0.15	-0.179002	- 0.178609	-0.178977
0.20	-0.235817	- 0.235816	-0.235785
0.25	-0.290167	- 0.290652	-0.290129
0.30	-0.341407	- 0.341406	-0.341365
0.35	-0.388872	- 0.387768	-0.388827
0.40	-0.431864	- 0.431863	-0.431817
0.45	-0.469644	- 0.474651	-0.469587

Source: Author's own elaboration.

The second case study proposes the following values of the parameters:

$$M = 0.5, Re = 6, K_1 = 0.005 \qquad (22)$$

Again, from the numerical solution for this set of parameters (equation 22) we get:

$$\begin{matrix} (0.1, -0.1618177) \\ (0.2, -0.3084747) \\ (0.3, -0.4246298) \\ (0.4, -0.4942297) \\ (0.5, -0.500000) \end{matrix} \qquad (23)$$

Next, we substitute equation 23 into equation 18, and also into the necessary derivatives of equation 18, aiming to obtain a set of nonlinear algebraic equations for the unknown A, B, C, D, E and F whose solutions provide the following approximate solution:

$$y(\eta) = -\frac{6218}{3773}\eta + \frac{37285}{52}\eta^9 - \frac{124107}{46}\eta^8 + \frac{491292}{157}\eta^7 - \frac{181582}{109}\eta^6 + \frac{38487}{85}\eta^5 - \frac{10425}{169}\eta^4 + \frac{12424}{2069}\eta^3 \quad (24)$$

Table 2 compares the exact solution of the proposed problem with parameters (equation 22) and approximated solution (equation 24). It is clear that equation 24 provides not only a handy solution but an accurate one.

Table 2. Comparison between exact solution and approximate solution (equation 24).

η	Exact Solution	BVPP (equation 24), this work
0.05	-0.08185511	-0.08191840
0.10	-0.16181779	-0.16181769
0.15	-0.23799245	-0.23807544
0.20	-0.30847479	-0.30847469
0.25	-0.37134120	-0.37111720
0.30	-0.42462983	-0.42462986
0.35	-0.46630931	-0.46695485
0.40	-0.49422978	-0.49423013
0.45	-0.50604904	-0.50297581

Source: Author's own elaboration.

Discussion

This work proposed the boundary value problems Picard method (BVPP) with the purpose of finding an explicit analytical approximate solution for the relevant problem of the magnetohydrodynamic flow of blood through a porous channel. The investigation on this problem is justified because the results obtained could be applicable to keep the blood flow at a pre-set level in a surgery, among many other applications (Misra *et al*., 2011). In fact, as far as we know, there are only two previous articles for this important problem. Misra *et al*. (2011) were the first to present a very detailed work from several points of view; their study presented a mathematical procedure focused on obtaining the numerical solution for the complicated problem (equations 2 and 3). The authors introduced a procedure based on an analysis of perturbation and after they employed a finite difference scheme, which presented good results. On the other hand, Vazquez-Leal (2020) introduced a precise analytical approximate solution for the same problem; this paper consists of the novel continuum-cancelation Leal-method (CCLM) aiming to get analytical approximate solutions for complicated nonlinear problems like this. Briefly, this method employs a process that involves the continuum cancelation of the residual error of multiple selected points. We have to emphasize that, unlike CCLM, the numerical solution gives a table of values (x, y) that describe the proposed problem, while CCLM provided an analytical solution from which it is possible to get one value of y, for any value of x in the domain of the problem. Nevertheless, from Vazquez-Leal (2020) it is clear that even the analytical solution obtained by CCLM, although it is precise, it is not handy. Actually, the solutions 17, 18, and 19 of the above mentioned article consist of rational functions with fourth-order polynomials numerators and fifth-order polynomial denominators. This, independently of the difficulty to utilize the CCLM algorithm, has led us to propose the boundary value problems Picard method as a valuable alternative in order to find handy analytical approximate solutions for the proposed problem.

Noticing that BVPP admits the proposal of a general trial function aimed to take advantage of the knowledge of a set of points of the sought solution, most of the times these points come from the numerical solution of the problem. The procedure is simple: to begin, the differential equation is rewritten in order to solve it as an iterative integral equation; next, we propose an initial approximation function which initially contains several unknown parameters to be determined; the iterative process starts from the afore mentioned initial function and generally a few iterations are required to obtain an accurate solution. At this point, the above mentioned known points are substituted in order to obtain a system of algebraic equations whose solution determines the constants of the proposed solution for BVPP. Just as it has occurred in this work, many times the first iteration of BVPP is enough to get a good solution.

In retrospective, the use of equation 17 as an initial function for the BVPP iteration with parameters calculated for the example in the first case study would have resulted in equation 21 after only one iteration. Given the precision of equation 21, it is correct to claim that the first iteration of the proposed method is accurate, and BVPP accelerates the convergence. Table 1 compares the BVPP solution of equation 21 for the values $M = 2$, $Re = 5$, $K_1 = 0.005$, with the numerical solution and solution 17 of CCLM method (Vazquez-Leal, 2020) for the same values of the above parameters. From this table, it is clear that BVPP solution 21 and CCLM solution are very competitive. Also, the proposed solution 21 is a handy expression that consists in a polynomial of degree seven of only six terms, unlike of the rational solution CCLM (equation 17) that contains a total of ten terms, which implies a greater computational effort. In the same way, Table 2 compares the BVPP solution of equation 24 for the values $M = 0.5$, $Re = 6$, and $K_1 = 0.005$ with the numeric solution of equations 2 and 3 for the same values of the parameters of equation 22. From this table, it is clear that BVPP solution of equation 24 is again a handy competitive polynomial of degree seven, which implies fewer computational efforts as well.

On the other hand, we note that the numerical solution employed for the first case study is not easy to obtain; therefore, the numerical solution for the problem given by equations 2 and 3, with study parameters provided in equation 19, required a special homotopy treatment (Diaz-Arango *et al*., 2018). Thus, the success of BVPP method consisted in providing an accurate analytical approximate solution for the numerical solution deduced from the homotopy technique (Diaz-Arango *et al*., 2018).

Conclusions

This work presented an accurate approximate solution for the highly nonlinear differential equation that describes the magnetohydrodynamic flow of blood through a porous channel by using the BVPP method. A relevant point of this method is its freedom to propose an adequate trial function. As it was already mentioned, the general strategy was to generate an explicit solution provided with a set of adjustment parameters which were evaluated from the knowledge of the numeric solution for the proposed problem. It is relevant to note that BVPP turns the problem of obtaining a solution for the very complicated non linear differential equation (see equation 2) into the solution of a nonlinear algebraic system, which is a known issue and it can be solved by several mathematical softwares. From the afore mentioned analysis, it is expected that this work contributes to break the paradigm that an effective method has to be necessarily long, cumbersome, and complicated. The proposal of BVPP method is that of an effective method which is easy to use for solving complicated highly nonlinear problems just like the one presented in this paper.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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